Statistical Language Models

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Statistical NLP

Broad multidisciplinary area

- Linguistics to provide models of language
- Psychology to provide models of cognitive processes
- Information theory to provide models of communication
- Mathematics & Statistics to provide tools to analyze and acquire such models
- Computer Science to implement computable models

Problems of the traditional approach (1)

- Language Acquisition:
  Children try and discard syntax rules progressively
- Language Change:
  Language changes along time (*ale* vs. *eel*, *while* as Adv vs. Noun, *near* as Prep vs. Adj)
- Language Variation:
  Dialect continuum (e.g. Inuit)
- Language is a collection of statistical distributions:
  Weights for rules (phonetic, syntactic, etc) change when learning, along time, between communities...

Problems of the traditional approach (2)

- Structural ambiguity

  Our company is training workers *Parker saw Mary*
  Our problem is training workers *The a are of I*
  Our product is training wheels

- Scalability: scaling up from small and domain specific applications
- Practicallity: Time costly to build systems with good coverage
- Brittleness: understanding metaphors
- Reasoning: Requires world knowledge and common sense knowledge $\Rightarrow$ learning

How Statistics helps

- Disambiguation: Stochastic grammars. *John walks*
- Degrees of grammaticality
- Naturalness: *strong tea, powerful car*
- Structural preferences:
  The emergency crews hate most is domestic violence
- Error tolerance:
  We sleeps *Thanks for all you help*
- Learning on the fly:
  One hectare is a hundred ares
  The are a of I
- Lexical Acquisition.

Zipf's Laws (1929)

- Word frequency is inversely proportional to its rank (speaker/hearer minimum effort) $f \sim 1/r$
- Number of senses is proportional to frequency root $m \sim \sqrt{f}$
- Frequency of intervals between repetitions is inversely proportional to the length of the interval $F \sim 1/I$
- Random generated languages satisfy Zipf's laws
- Frequency based approaches are hard, since most words are rare
  - Most common 5% words account for about 50% of a text
  - 90% least common words account for less than 10% of the text
  - Almost half of the words in a text occurr only once

Usual Objections

Stochastic models are for engineers, not for scientists

- Approximation to handle information impractical to collect in cases where initial conditions cannot be exactly determined (e.g. as queue theory models dynamical systems).
- If the system is not deterministic (i.e. has *emergent* properties), an stochastic account is more insightful than a reductionistic approach (e.g. statistical mechanics)

Chomsky's heritage: Statistics can not capture NL structure

- Techniques to estimate probabilities of unseen events.
- Chomsky's criticisms can be applied to Finite State, N-gram or Markov models, but not to all stochastic models.

Conclusions

- Statistical methods are relevant to language acquisition, change, variation, generation and comprehension.
- Pure algebraic methods are inadequate for understanding many important properties of language, such as the measure of goodness that allows to identify the correct parse among a large candidate set.
- The focus of computational linguistics has been up to now on technology, but the same techniques promise progress at unanswered questions about the nature of language.

Basics

- Random variable: Function on a stochastic process.
  $X : \Omega \longrightarrow \mathcal{R}$
- Continuous and discrete random variables.
- Probability mass (or density) function, Frequency function:
  $p(x) = P(X = x)$.
  Discrete R.V.: $\sum_x p(x) = 1$
  Continuous R.V: $\int_{-\infty}^{\infty} p(x)dx = 1$
- Distribution function: $F(x) = P(X \leq x)$
- Expectation and variance, standard deviation
  $E(X) = \mu = \sum_x xp(x)$
  $VAR(X) = \sigma^2 = E((X - E(X))^2) = \sum_x (x - \mu)^2 p(x)$

Joint and Conditional Distributions

- Joint probability mass function: $p(x,y)$
- Marginal distribution:

$$p_X(x) = \sum_y p(x,y) \qquad p_{X|Y}(x \mid y) = \frac{p(x,y)}{p_Y(y)}$$
$$p_Y(y) = \sum_x p(x,y)$$

Simplified Polynesian. Sequences of C-V syllabes: Two random variables C,V

P(C,V)	p	t	k	
a	1/16	3/8	1/16	1/2
i	1/16	3/16	0	1/4
u	0	3/16	1/16	1/4
	1/8	3/4	1/8	

$P(p \mid i) = ?$
$P(a \mid t \vee k) = ?$
$P(a \vee i \mid p) = ?$

Samples and Estimators

- Random samples
- Sample variables:

  Sample mean: $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

  Sample variance: $s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{\mu}_n)^2$.
- Law of Large Numbers: as n increases, $\bar{\mu}_n$ and s_n^2 converge to μ and σ^2
- Estimators: Sample variables used to estimate real parameters.

Finding good estimators: MLE

Maximum Likelihood Estimation (MLE)

- Choose the alternative that maximizes the probability of the observed outcome.
- $\bar{\mu}_n$ is a MLE for $E(X)$
- s_n^2 is a MLE for σ^2
- Data sparseness problem. Smoothing tecnhiques.

$P(a,b)$	dans	en	à	sur	au-cours-de	pendant	selon	
in	0.04	0.10	0.15	0	0.08	0.03	0	0.40
on	0.06	0.25	0.10	0.15	0	0	0.04	0.60
total	0.10	0.35	0.25	0.15	0.08	0.03	0.04	1.0

Finding good estimators: MEE

Maximum Entropy Estimation (MEE)

- Choose the alternative that maximizes the entropy of the obtained distribution, maintaning the observed probabilities.

Observations:
$p(en \vee \grave{a}) = 0.6$

$P(a,b)$	dans	en	à	sur	au-cours-de	pendant	selon	
in	0.04	0.15	0.15	0.04	0.04	0.04	0.04	
on	0.04	0.15	0.15	0.04	0.04	0.04	0.04	
total		0.6 (en + à)						1.0

Finding good estimators: MEE

Maximum Entropy Estimation (MEE)

- Choose the alternative that maximizes the entropy of the obtained distribution, maintaning the observed probabilities.

Observations:

$p(en \vee à) = 0.6; \qquad p((en \vee à) \wedge in) = 0.4$

$P(a,b)$	dans	en	à	sur	au-cours-de	pendant	selon	
in	0.04	**0.20**	**0.20**	0.04	0.04	0.04	0.04	
on	0.04	0.10	0.10	0.04	0.04	0.04	0.04	
total		0.6 (en + à)						1.0

Finding good estimators: MEE

Maximum Entropy Estimation (MEE)

- Choose the alternative that maximizes the entropy of the obtained distribution, maintaning the observed probabilities.

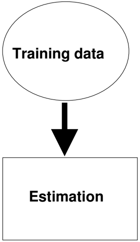
Observations:

$p(en \vee \text{à}) = 0.6; \qquad p((en \vee \text{à}) \wedge in) = 0.4; \qquad p(in) = 0.5$

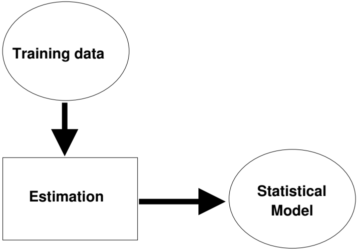
$P(a,b)$	dans	en	à	sur	au-cours-de	pendant	selon	
in	0.02	**0.20**	**0.20**	0.02	0.02	0.02	0.02	**0.5**
on	0.06	0.10	0.10	0.06	0.06	0.06	0.06	
total		**0.6** (en + à)						1.0

Statistical models for NLP

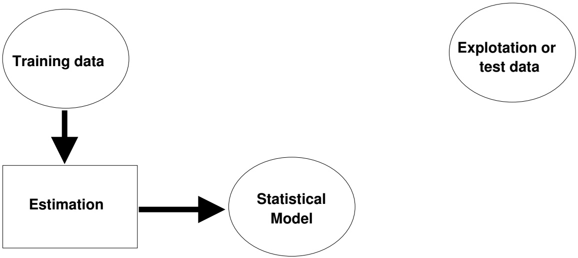
Statistical models for NLP



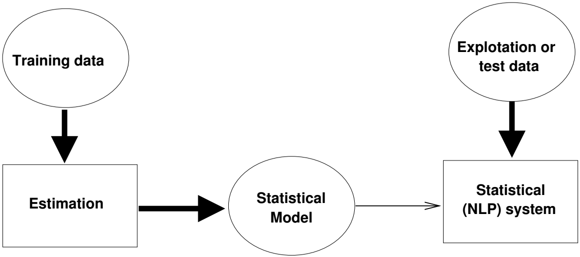
Statistical models for NLP



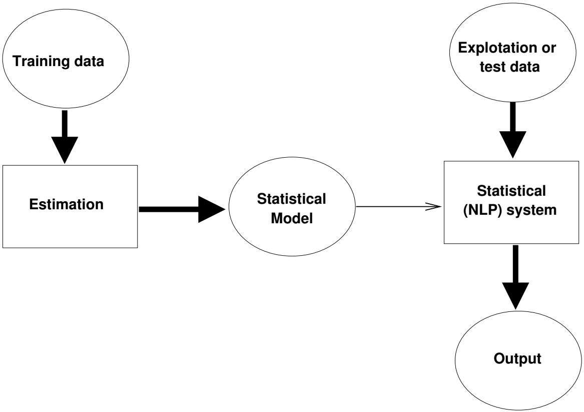
Statistical models for NLP



Statistical models for NLP



Statistical models for NLP



Prediction Models & Similarity Models

- Prediction Models: Able to *predict* probabilities of future events, knowing past and present.
- Similarity Models: Able to compute *similarities* between objects (may be used to predict, EBL).

Similarity Models

- Objects represented as feature-vectors, feature-sets, distribution-vectors.
- Used to group objects (clustering, data analysis, pattern discovery, ...)
- If existig objects are classified, similarity may be used as a prediction (example-based ML techniques).
- Example: Document representation
  - Documents are represented as vectors in a high dimensional $\mathbb{R}^n$ space.
  - Dimensions are word forms, lemmas, NEs, n-grams, ...
  - Values may be either binary or real–valued (count, frequency, ...)
  - Vector space algebra and metrics can be used

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \qquad \vec{x}^T = [x_1 \dots x_N] \qquad |\vec{x}| = \sqrt{\sum_{i=1}^{N} x_i^2}$$

Prediction Models

Example: Noisy Channel Model (Shannon 48)

NLP Applications

Appl.	Input	Output	$p(i)$	$p(o \mid i)$
MT	L word sequence	M word sequence	$p(L)$	Translation model
OCR	Actual text	Text with mistakes	prob. of language text	model of OCR errors
PoS tagging	PoS tags sequence	word sequence	prob. of PoS sequence	$p(w \mid t)$
Speech recog.	word sequence	speech signal	prob. of word sequence	acoustic model

Given $\mathbf{o}$, we want to find the most likely $\mathbf{i}$

$$\operatorname*{argmax}_{\mathbf{i}} \Pr(\mathbf{i} \mid \mathbf{o}) = \operatorname*{argmax}_{\mathbf{i}} \Pr(\mathbf{o}, \mathbf{i}) = \operatorname*{argmax}_{\mathbf{i}} \Pr(\mathbf{i}) \Pr(\mathbf{o} \mid \mathbf{i})$$

Inference & Modeling

- Using data to infer information about distributions
  - Parametric / non-parametric estimation
  - Finding good estimators: MLE, MEE, ...
- Example: Language Modeling (Shannon game), N-gram models.
- Predictions based on past behaviour
  - Target / classification features $\rightarrow$ Independence assumptions
  - Equivalence classes (bins).
    Granularity: discrimination *vs.* statistical reliability

N-gram models

- Predicting the next word in a sequence, given the *history* or *context*. $P(w_n \mid w_1 \dots w_{n-1})$
- Markov assumption: Only *local* context (of size $n-1$) is taken into account. $P(w_i \mid w_{i-n+1} \dots w_{i-1})$
- bigrams, trigrams, four-grams ($n = 2, 3, 4$). *Sue swallowed the large green* <?>
- Parameter estimation (number of equivalence classes)
- Parameter reduction: stemming, semantic classes, PoS, ...

Model	Parameters
bigram	$20,000^2 = 4 \times 10^8$
trigram	$20,000^3 = 8 \times 10^{12}$
four-gram	$20,000^4 = 1.6 \times 10^{17}$

Language model sizes for a 20,000 words vocabulary

MLE Overview

Estimate the probability of the target feature based on observed data. The prediction task can be reduced to having good estimations of the *n*-gram distribution:

$$P(w_n \mid w_1 \ldots w_{n-1}) = \frac{P(w_1 \ldots w_n)}{P(w_1 \ldots w_{n-1})}$$

- **MLE (Maximum Likelihood Estimation)**

  $P_{MLE}(w_1 \ldots w_n) = \frac{C(w_1 \ldots w_n)}{N}$

  $P_{MLE}(w_n \mid w_1 \ldots w_{n-1}) = \frac{C(w_1 \ldots w_n)}{C(w_1 \ldots w_{n-1})}$

  - No probability mass for unseen events
  - Unsuitable for NLP
  - Data sparseness, Zipf's Law

Notation

- $C(w_1 \dots w_n)$: Observed occurrence count for n-gram $w_1 \dots w_n$.
- $C_A(w_1 \dots w_n)$: Observed occurrence count for n-gram $w_1 \dots w_n$ on data subset A.
- N: Number of observed n-gram occurrences

$$N = \sum_{w_1 \dots w_n} C(w_1 \dots w_n)$$

- N_k: Number of classes (n-grams) observed k times.
- N_k^A: Number of classes (n-grams) observed k times on data subset A.
- B: Number of equivalence classes or bins (number of potentially observable n-grams).

Smoothing 1 - Adding Counts

- **Laplace's Law** (adding one)

  $$P_{LAP}(w_1 \dots w_n) = \frac{C(w_1 \dots w_n) + 1}{N + B}$$

  - For large values of B too much probability mass is assigned to unseen events

- **Lidstone's Law**

  $$P_{LID}(w_1 \dots w_n) = \frac{C(w_1 \dots w_n) + \lambda}{N + B\lambda}$$

  - Usually $\lambda = 0.5$, *Expected Likelihood Estimation*.
  - Equivalent to linear interpolation between MLE and uniform prior, with $\mu = N/(N + B\lambda)$,

    $$P_{LID}(w_1 \dots w_n) = \mu\frac{C(w_1 \dots w_n)}{N} + (1 - \mu)\frac{1}{B}$$

Smoothing 2 - Discounting Counts

- **Absolute Discounting**

$$P_{ABS}(w_1 \ldots w_n) = \begin{cases} \frac{r-\delta}{N} & \text{if } r > 0 \\ \frac{(B-N_0)\delta/N_0}{N} & \text{otherwise} \end{cases}$$

- **Linear Discounting**

$$P_{LIN}(w_1 \ldots w_n) = \begin{cases} \frac{(1-\alpha)r}{N} & \text{if } r > 0 \\ \frac{\alpha}{N_0} & \text{otherwise} \end{cases}$$

Smoothing 3 - Held Out Data

- *Notation:* γ stands for $w_1 \ldots w_n$.
- Divide the train corpus in two subsets, A and B.
- Define: $T_r^{AB} = \sum_{\gamma : C_A(\gamma)=r} C_B(\gamma)$
- **Held Out Estimator**

$$P_{HO}(w_1 \ldots w_n) = \frac{T^{AB}_{C_A(\gamma)}}{N^A_{C_A(\gamma)}} \times \frac{1}{N}$$

- **Cross Validation** (deleted estimation)

$$P_{DEL}(w_1 \ldots w_n) = \frac{T^{AB}_{C_A(\gamma)} + T^{BA}_{C_B(\gamma)}}{N^A_{C_A(\gamma)} + N^B_{C_B(\gamma)}} \times \frac{1}{N}$$

- **Cross Validation** (Leave-one-out)

Combining Estimators

- **Simple Linear Interpolation**

  $$P_{LI}(w_n \mid w_{n-2}, w_{n-1}) = \\ = \lambda_1 P_1(w_n) + \lambda_2 P_2(w_n \mid w_{n-1}) + \lambda_3 P_3(w_n \mid w_{n-2}, w_{n-1})$$

- **General Linear Interpolation**

$$P_{LI}(w_n \mid h) = \sum_{i=1}^{k} \lambda_i(h) P_i(w \mid h_i)$$

- **Katz's Backing-off**

$$P_{BO}(w_i \mid w_{i-n+1} \dots w_{i-1}) = \begin{cases} (1 - d_{w_{i-n+1} \dots w_{i-1}}) \dfrac{C(w_{i-n+1} \dots w_i)}{C(w_{i-n+1} \dots w_{i-1})} & \text{if } C(w_{i-n+1} \dots w_i) > k \\ \alpha_{w_{i-n+1} \dots w_{i-1}} P_{BO}(w_i \mid w_{i-n+2} \dots w_{i-1}) & \text{otherwise} \end{cases}$$

MEM Overview

- Maximum Entropy: alternative estimation technique.
- Able to deal with different kinds of evidence
- ME principle:
  - Do not assume anything about non-observed events.
  - Find the most uniform (maximum entropy, less informed) probability distribution that matches the observations.
- Example:

$p(a,b)$	0	1	
x	?	?	
y	?	?	
total	0.6		1.0

Observations

$p(a,b)$	0	1	
x	0.5	0.1	
y	0.1	0.3	
total	0.6		1.0

One possible $p(a,b)$

$p(a,b)$	0	1	
x	0.3	0.2	
y	0.3	0.2	
total	0.6		1.0

Max.Entropy $p(a,b)$

ME Modeling

- Observed facts are constraints for the desired model p.
- Constraints take the form of feature functions:

$$f_i : \varepsilon \rightarrow \{0, 1\}$$

- The desired model must satisfy the constraints:

$$E_p(f_i) = E_{\tilde{p}}(f_i) \quad \forall i$$

where:

$$E_p(f_i) = \sum_{x \in \varepsilon} p(x) f_i(x) \quad \text{expectation of model } p.$$

$$E_{\tilde{p}}(f_i) = \sum_{x \in \varepsilon} \tilde{p}(x) f_i(x) \quad \text{observed expectation.}$$

Example

- Example:

$$\varepsilon = \{x, y\} \times \{0, 1\}$$

$p(a,b)$	0	1	
x	?	?	
y	?	?	
total	0.6		1.0

- Observed fact: $p(x,0) + p(y,0) = 0.6$
- Encoded as a constraint: $E_p(f_1) = 0.6$
  where:
  - $f_1(a,b) = \begin{cases} 1 & \textit{if } b = 0 \\ 0 & \textit{otherwise} \end{cases}$
  - $E_p(f_1) = \sum_{(a,b)\in\{x,y\}\times\{0,1\}} p(a,b) f_1(a,b)$

Probability Model

- There is an infinite set P of probability models consistent with observations:

$$P = \{p \mid E_p(f_i) = E_{\tilde{p}}(f_i), \ \forall i = 1 \ldots k\}$$

- Maximum entropy model

$$p^* = \operatorname*{argmax}_{p \in P} H(p)$$

$$H(p) = -\sum_{x \in \varepsilon} p(x) \log p(x)$$

Conditional Probability Model

- For NLP applications, we are usually interested in conditional distributions $P(A|B)$, thus:

$$E_{\tilde{p}}(f_j) = \sum_{a,b} \tilde{p}(a,b) f_j(a,b)$$

$$E_p(f_j) = \sum_{a,b} \tilde{p}(b) p(a \mid b) f_j(a,b)$$

- Maximum entropy model

$$p^* = \operatorname*{argmax}_{p \in P} H(p)$$

$$H(p) = H(A \mid B) = -\sum_{a,b} \tilde{p}(b) p(a \mid b) \log p(a \mid b)$$

Parameter Estimation

Example: Maximum entropy model for translating *in* to French

- No constraints

$P(x)$	dans	en	à	au-cours-de	pendant	
	0.2	0.2	0.2	0.2	0.2	
total						1.0

- With constraint $p(dans) + p(en) = 0.3$

$P(x)$	dans	en	à	au-cours-de	pendant	
	0.15	0.15	0.233	0.233	0.233	
total	**0.3**					1.0

- With constraints $p(dans) + p(en) = 0.3;\ p(en) + p(à) = 0.5$

...Not so easy !

Parameter estimation

- Exponential models. (Lagrange multipliers optimization)
  $p(a \mid b) = \frac{1}{Z(b)} \prod_{j=1}^{k} \alpha_j^{f_j(a,b)} \qquad \alpha_j > 0$
  $Z(b) = \sum_a \prod_{i=1}^{k} \alpha_i^{f_i(a,b)}$
- also formuled as
  $p(a \mid b) = \frac{1}{Z(b)} \exp(\sum_{j=1}^{k} \lambda_j f_j(a, b))$
  $\lambda_i = \ln \alpha_i$
- Each model parameter weights the influence of a feature.
- Optimal parameters (ME model) can be computed with:
  - GIS. Generalized Iterative Scaling(Darroch & Ratcliff 72)
  - IIS. Improved Iterative Scaling (Della Pietra et al. 96)
  - LM-BFGS. Limited Memory BFGS (Malouf 03)

Improved Iterative Scaling (IIS)

Input: Feature functions $f_1 \ldots f_n$, empirical distribution $\widetilde{p}(a, b)$
Output: λ_i^* parameters for optimal model p^*

Start with $\lambda_i = 0$ for all $i \in \{1 \ldots n\}$
Repeat
For each $i \in \{1 \ldots n\}$ **do**
let $\Delta\lambda_i$ be the solution to

$$\sum_{a,b} \widetilde{p}(b) p(a \mid b) f_i(a, b) \exp(\Delta\lambda_i \sum_{j=1}^{n} f_j(a, b)) = \widetilde{p}(f_i)$$

$\lambda_i \leftarrow \lambda_i + \Delta\lambda_i$
end for
Until all λ_i have converged

Application to NLP Tasks

- Speech processing (Rosenfeld 94)
- Machine Translation (Brown et al 90)
- Morphology (Della Pietra et al. 95)
- Clause boundary detection (Reynar & Ratnaparkhi 97)
- PP-attachment (Ratnaparkhi et al 94)
- PoS Tagging (Ratnaparkhi 96, Black et al 99)
- Partial Parsing (Skut & Brants 98)
- Full Parsing (Ratnaparkhi 97, Ratnaparkhi 99)
- Text Categorization (Nigam et al 99)

PoS Tagging (Ratnaparkhi 96)

- Probabilistic model over $H \times T$

$$h_i = (w_i, w_{i+1}, w_{i+2}, w_{i-1}, w_{i-2}, t_{i-1}, t_{i-2})$$

$$f_j(h_i, t) = \begin{cases} 1 & \textit{if suffix}(w_i) = \texttt{ing} \wedge t = \texttt{VBG} \\ 0 & \textit{otherwise} \end{cases}$$

- Compute $p^*(h, t)$ using GIS
- Disambiguation algorithm: *beam search*

$$p(t \mid h) = \frac{p(h, t)}{\sum_{t' \in T} p(h, t')}$$

$$p(t_1 \dots t_n \mid w_1 \dots w_n) = \prod_{i=1}^{n} p(t_i \mid h_i)$$

Text Categorization (Nigam et al 99)

- Probabilistic model over $W \times C$

$$d = (w_1, w_2 \ldots w_N)$$

$$f_{w,c'}(d,c) = \begin{cases} \frac{N(d,w)}{N(d)} & if\ c = c' \\ 0 & otherwise \end{cases}$$

- Compute $p^*(c \mid d)$ using IIS
- Disambiguation algorithm: Select class with highest

$$P(c \mid d) = \frac{1}{Z(d)} exp(\sum_i \lambda_i f_i(d,c))$$

MEM Summary

- Advantages
  - Teoretically well founded
  - Enables combination of random context features
  - Better probabilistic models than MLE (no smoothing needed)
  - General approach (features, events and classes)
- Disadvantages
  - Implicit probabilistic model (joint or conditional probability distribution obtained from model parameters).
  - High computational cost of GIS and IIS.
  - Overfitting in some cases.

Graphical Models

- **Generative models:**
  - Bayes rule ⇒ independence assumptions.
  - Able to *generate* data.
- **Conditional models:**
  - No independence assumptions.
  - Unable to generate data.

Most algorithms of both kinds make assumptions about the nature of the data-generating process, predefining a fixed model structure and only acquiring from data the distributional information.

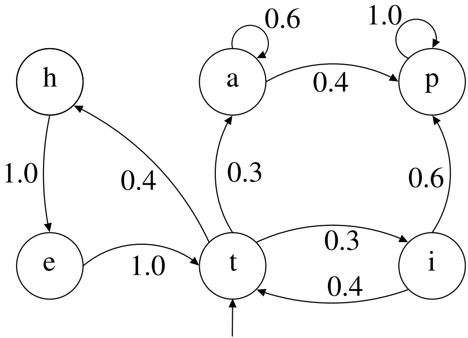
Usual Statistical Models in NLP

- **Generative models:**
  - Graphical: HMM (Rabiner 1990), IOHMM (Bengio 1996). Automata-learning algorithms: *No assumptions about model structure*. VLMM (Rissanen 1983), Suffix Trees (Galil & Giancarlo 1988), CSSR (Shalizi & Shalizi 2004).
  - Non-graphical: Stochastic Grammars (Lary & Young 1990)
- **Conditional models:**
  - Graphical: discriminative MM (Bottou 1991), MEMM (McCallum et al. 2000), CRF (Lafferty et al. 2001).
  - Non-graphical: Maximum Entropy Models (Berger et al 1996).

[Visible] Markov Models

- $X = (X_1, \ldots, X_T)$ sequence of random variables taking values in $S = \{s_1, \ldots, s_N\}$
- Markov Properties
  - Limited Horizon:
    $P(X_{t+1} = s_k \mid X_1, \ldots, X_t) = P(X_{t+1} = s_k \mid X_t)$
  - Time Invariant (Stationary):
    $P(X_{t+1} = s_k \mid X_t) = P(X_2 = s_k \mid X_1)$
- Transition matrix:
  $a_{ij} = P(X_{t+1} = s_j \mid X_t = s_i);\quad a_{ij} \geq 0,\ \forall i, j;\ \sum_{j=1}^{N} a_{ij} = 1,\ \forall i$
- Initial probabilities (or extra state s_0):
  $\pi_i = P(X_1 = s_i);\quad \sum_{i=1}^{N} \pi_i = 1$

MM Example



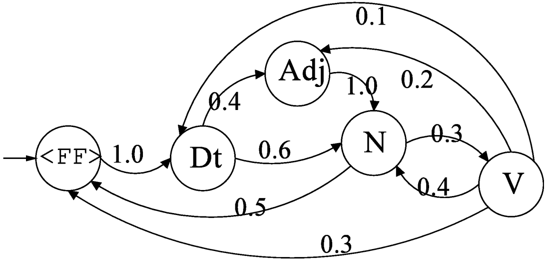
Sequence probability:

$$
\begin{aligned}
P(X_1, .., X_T) = \\
&= P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_1 X_2) \ldots P(X_T \mid X_1..X_{T-1}) \\
&= P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2) \ldots P(X_T \mid X_{T-1}) \\
&= \pi_{X_1} \prod_{t=1}^{T-1} a_{X_t X_{t+1}}
\end{aligned}
$$

Hidden Markov Models (HMM)

- States and Observations
- Emission Probability:
  $$b_{ik} = P(O_t = k \mid X_t = s_i)$$
- Used when underlying events probabilistically generate surface events:
  - PoS tagging (hidden states: PoS tags, observations: words)
  - ASR (hidden states: phonemes, observations: sound)
  - ...
- Trainable with unannotated data. Expectation Maximization (EM) algorithm.
- arc-emission *vs* state-emission

Example: PoS Tagging



Emission probabilities	.	the	this	cat	kid	eats	runs	fish	fresh	little	big
<FF>	1.0										
Dt		0.6	0.4								
N				0.6	0.1			0.3			
V						0.7	0.3				
Adj									0.3	0.3	0.4

HMM Fundamental Questions

Q1. **Observation probability (decoding):** Given a model $\mu = (A, B, \pi)$, how we do efficiently compute how likely is a certain observation ? That is, $P_\mu(O)$

Q2. **Classification:** Given an observed sequence O and a model μ, how do we choose the state sequence $(X_1, \ldots, X_T)$ that best explains the observations?

Q3. **Parameter estimation:** Given an observed sequence O and a space of possible models, each with different parameters (A, B, π), how do we find the model that best explains the observed data?

Question 1. Observation probability

- Let $O = (o_1, \ldots, o_T)$ observation sequence.
- For any state sequence $X = (X_1, \ldots, X_T)$, we have:

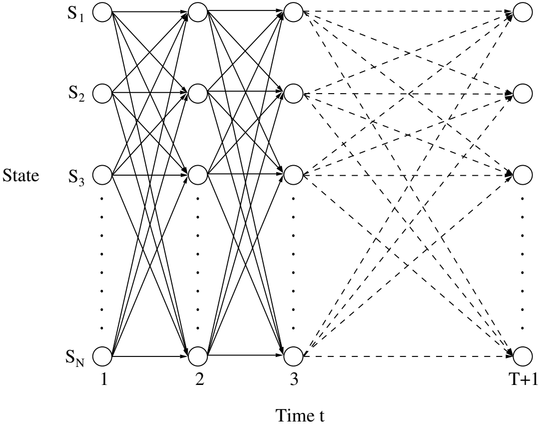
$$
\begin{aligned}
P_\mu(O \mid X) &= \prod_{t=1}^{T} P_\mu(o_t \mid X_t) \\
&= b_{X_1 o_1} b_{X_2 o_2} \ldots b_{X_T o_T}
\end{aligned}
$$

- $P_\mu(X) = \pi_{X_1} a_{X_1 X_2} a_{X_2 X_3} \ldots a_{X_{T-1} X_T}$
- $P_\mu(O) = \sum_X P_\mu(O, X) = \sum_X P_\mu(O \mid X) P_\mu(X)$

$$
= \sum_{X_1 \ldots X_T} \pi_{X_1} b_{X_1 o_1} \prod_{t=2}^{T} a_{X_{t-1} X_t} b_{X_t o_t}
$$

- Complexity: $\mathcal{O}(TN^T)$
- Dynammic Programming: Trellis/lattice. $\mathcal{O}(TN^2)$

Trellis



Fully connected HMM where one can move from any state to any other at each step. A node $\{s_i, t\}$ of the trellis stores information about state sequences which include $X_t = i$.

Forward & Backward computation

Forward procedure $\mathcal{O}(TN^2)$

We store $\alpha_i(t)$ at each trellis node $\{s_i, t\}$.

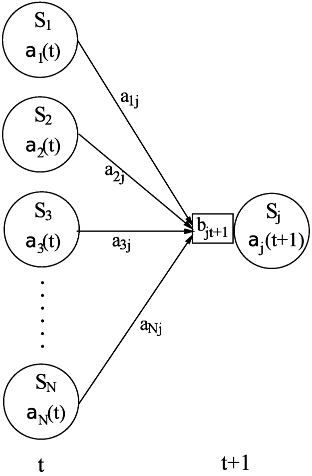
$\alpha_i(t) = P_\mu(o_1 \dots o_t, X_t = i)$ Probability of emmiting $o_1 \dots o_t$ and reach state s_i at time t.

1. Inicialization: $\alpha_i(1) = \pi_i b_{io_1}; \quad \forall i = 1 \dots N$
2. Induction: $\forall t : 1 \leq t < T$

$$\alpha_j(t+1) = \sum_{i=1}^{N} \alpha_i(t) a_{ij} b_{jo_{t+1}}; \quad \forall j = 1 \dots N$$

3. Total: $P_\mu(O) = \sum_{i=1}^{N} \alpha_i(T)$

Forward computation



Closeup of the computation of forward probabilities at one node. The forward probability $\alpha_j(t+1)$ is calculated by summing the product of the probabilities on each incoming arc with the forward probability of the originating node.

Forward & Backward computation

Backward procedure $\mathcal{O}(TN^2)$

We store $\beta_i(t)$ at each trellis node $\{s_i, t\}$.

$$\beta_i(t) = P_\mu(o_{t+1} \dots o_T \mid X_t = i)$$

Probability of emmiting $o_{t+1} \dots o_T$ given we are in state s_i at time t.

1. Inicialization: $\beta_i(T) = 1 \qquad \forall i = 1 \dots N$
2. Induction: $\forall t : 1 \leq t < T$
$$\beta_i(t) = \sum_{j=1}^{N} a_{ij} b_{j o_{t+1}} \beta_j(t+1) \qquad \forall i = 1 \dots N$$
3. Total: $P_\mu(O) = \sum_{i=1}^{N} \pi_i b_{i o_1} \beta_i(1)$

Forward & Backward computation

Combination

$$\begin{aligned} P_\mu(O, X_t = i) &= P_\mu(o_1 \ldots o_{t-1}, X_t = i, o_t \ldots o_T) \\ &= \alpha_i(t)\beta_i(t) \end{aligned}$$

$$P_\mu(O) = \sum_{i=1}^{N} \alpha_i(t)\beta_i(t) \qquad \forall t : 1 \leq t \leq T$$

Forward and Backward procedures are particular cases of this equation when $t = 1$ and $t = T$ respectively.

Question 2. Best state sequence

- Most likely path for a given observation O:

$$\begin{aligned}\operatorname*{argmax}_X P_\mu(X \mid O) &= \operatorname*{argmax}_X \frac{P_\mu(X, O)}{P_\mu(O)} \\ &= \operatorname*{argmax}_X P_\mu(X, O) \quad \text{(since } O \text{ is fixed)}\end{aligned}$$

- Compute the best sequence with the same recursive approach than in FB: Viterbi algorithm, $\mathcal{O}(TN^2)$.

- $\delta_j(t) = \max_{X_1 \ldots X_{t-1}} P_\mu(X_1 \ldots X_{t-1} s_j, o_1 \ldots o_t)$

  Highest probability of any sequence reaching state s_j at time t after emmitting $o_1 \ldots o_t$

- $\psi_j(t) = last(\operatorname*{argmax}_{X_1 \ldots X_{t-1}} P_\mu(X_1 \ldots X_{t-1} s_j, o_1 \ldots o_t))$

  Last state (X_{t-1}) in highest probability sequence reaching state s_j at time t after emmitting $o_1 \ldots o_t$

Viterbi algorithm

1. Initialization: $\forall j = 1 \dots N$
   $\delta_j(1) = \pi_j b_{jo_1}$
   $\psi_j(1) = 0$
2. Induction: $\forall t : 1 \leq t < T$
   $\delta_j(t+1) = \max_{1 \leq i \leq N} \delta_i(t) a_{ij} b_{jo_{t+1}} \quad \forall j = 1 \dots N$
   $\psi_j(t+1) = \operatorname{argmax}_{1 \leq i \leq N} \delta_i(t) a_{ij} \quad \forall j = 1 \dots N$
3. Termination: backwards path readout.
   - $\hat{X}_T = \operatorname{argmax}_{1 \leq i \leq N} \delta_i(T)$
   - $\hat{X}_t = \psi_{\hat{X}_{t+1}}(t+1)$
   - $P(\hat{X}) = \max_{1 \leq i \leq N} \delta_i(T)$

Question 3. Parameter Estimation

Obtain model parameters (A, B, π) for the model μ that maximizes the probability of given observation O:

$$(A, B, \pi) = \operatorname*{argmax}_{\mu} P_{\mu}(O)$$

Baum-Welch algorithm

- Baum-Welch algorithm (*aka* Forward-Backward):
  1. Start with an initial model μ_0 (uniform, random, MLE...)
  2. Compute observation probability (F&B computation) using current model μ.
  3. Use obtained probabilities as data to reestimate the model, computing $\hat{\mu}$
  4. Let $\mu = \hat{\mu}$ and repeat until no significant improvement.
- Iterative hill-climbing: Local maxima.
- Particular application of Expectation Maximization (EM) algorithm.
- EM Property: $P_{\hat{\mu}}(O) \geq P_{\mu}(O)$

Definitions

- $\gamma_i(t) = P_\mu(X_t = i \mid O) = \dfrac{P_\mu(X_t = i, O)}{P_\mu(O)} = \dfrac{\alpha_i(t)\beta_i(t)}{\sum_{k=1}^{N} \alpha_k(t)\beta_k(t)}$

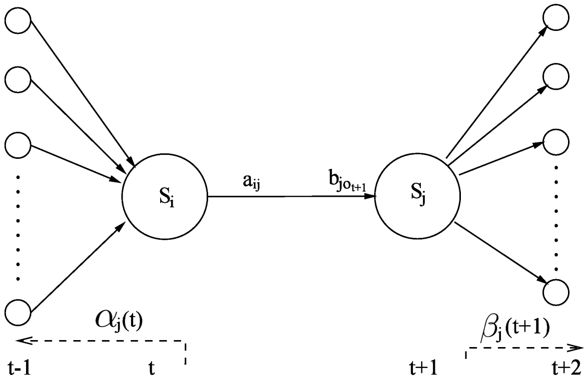
  Probability of being at state s_i at time t given observation O.

- $\varphi_t(i,j) = P_\mu(X_t = i, X_{t+1} = j \mid O) = \dfrac{P_\mu(X_t = i, X_{t+1} = j, O)}{P_\mu(O)}$

  $= \dfrac{\alpha_i(t) a_{ij} b_{jo_{t+1}} \beta_j(t+1)}{\sum_{k=1}^{N} \alpha_k(t)\beta_k(t)}$ probability of moving from state s_i at time t to state s_j at time $t+1$, given observation sequence O. Note that $\gamma_i(t) = \sum_{j=1}^{N} \varphi_t(i,j)$

$\sum_{t=1}^{T-1} \gamma_i(t)$ Expected number of transitions from state s_i in O.	$\sum_{t=1}^{T-1} \varphi_t(i,j)$ Expected number of transitions from state s_i to s_j in O.

Arc probability



Given an observation O, the model μ Probability $\varphi_t(i,j)$ of moving from state s_i at time t to state s_j at time $t+1$ given observation O.

Reestimation

Iterative reestimation

$$\hat{\pi}_i = \text{Expected frequency in state } s_i \text{ at time } (t=1) = \gamma_i(1)$$

$$\hat{a}_{ij} = \frac{\text{Expected number of transitions from } s_i \text{ to } s_j}{\text{Expected number of transitions from } s_i} = \frac{\sum_{t=1}^{T-1} \varphi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_i(t)}$$

$$\hat{b}_{jk} = \frac{\text{Expected number of emissions of } k \text{ from } s_j}{\text{Expected number of visits to } s_j} = \frac{\sum_{\{t:\, 1 \le t \le T,\, o_t = k\}} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$$

The Concept of Similarity

- *Similarity, proximity, affinity, distance, difference, divergence*
- We use *distance* when metric properties hold:
  - $d(x, x) = 0$
  - $d(x, y) \geq 0$ when $x \neq y$
  - $d(x, y) = d(y, x)$ (simmetry)
  - $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequation)
- We use *similarity* in the general case
  - Function: $sim : A \times B \rightarrow S$ (where S is often $[0, 1]$)
  - Homogeneous: $sim : A \times A \rightarrow S$ (e.g. word-to-word)
  - Heterogeneous: $sim : A \times B \rightarrow S$ (e.g. word-to-document)
  - Not necessarily symmetric, or holding triangular inequation.

The Concept of Similarity

- If A is a metric space, the distance in A may be used.
  - $D_{euclidean}(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| = \sqrt{\sum_i (x_i - y_i)^2}$
- Similarity *vs* distance
  - $sim_D(A, B) = \frac{1}{1+D(A,B)}$
  - monotonic: $min\{sim(x, y), sim(x, z)\} \geq sim(x, y \cup z)$

Applications

- Clustering, case-based reasoning, IR, ...
- Discovering related words - Distributional similarity
- Resolving syntactic ambiguity - Taxonomic similarity
- Resolving semantic ambiguity - Ontological similarity
- Acquiring selectional restrictions/preferences

Relevant Information

- Content (information about compared units)
  - Words: form, morphology, PoS, ...
  - Senses: synset, topic, domain, ...
  - Syntax: parse trees, syntactic roles, ...
  - Documents: words, collocations, NEs, ...
- Context (information about the situation in which simmilarity is computed)
  - Window–based *vs.* Syntactic–based
- External Knowledge
  - Monolingual/bilingual dictionaries, ontologies, corpora

Vectorial methods (1)

- L_1 norm, Manhattan distance, taxi-cab distance, city-block distance

$$L_1(\vec{x},\vec{y}) = \sum_{i=1}^{N} |x_i - y_i|$$

- L_2 norm, Euclidean distance

$$L_2(\vec{x},\vec{y}) = |\vec{x}-\vec{y}| = \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2}$$

- Cosine distance

$$cos(\vec{x},\vec{y}) = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}|\cdot|\vec{y}|} = \frac{\sum_i x_i y_i}{\sqrt{\sum_i x_i^2}\cdot\sqrt{\sum_i y_i^2}}$$

Vectorial methods (2)

- L_1 and L_2 norms are particular cases of Minkowsky measure

$$D_{minkowsky}(\vec{x},\vec{y}) = L_r(\vec{x},\vec{y}) = \left(\sum_{i=1}^{N}(x_i - y_i)^r\right)^{\frac{1}{r}}$$

- Camberra distance

$$D_{camberra}(\vec{x},\vec{y}) = \sum_{i=1}^{N}\frac{|x_i - y_i|}{|x_i + y_i|}$$

- Chebychev distance

$$D_{chebychev}(\vec{x},\vec{y}) = \max_i |x_i - y_i|$$

Set-oriented methods (3): Binary–valued vectors seen as sets

- Dice. $S_{dice}(X,Y) = \dfrac{2 \cdot |X \cap Y|}{|X| + |Y|}$
- Jaccard. $S_{jaccard}(X,Y) = \dfrac{|X \cap Y|}{|X \cup Y|}$
- Overlap. $S_{overlap}(X,Y) = \dfrac{|X \cap Y|}{\min(|X|,|Y|)}$
- Cosine. $cos(X,Y) = \dfrac{|X \cap Y|}{\sqrt{|X| \cdot |Y|}}$

Above similarities are in $[0,1]$ and can be used as distances simply substracting: $D = 1 - S$

Set-oriented methods (4): Agreement contingency table

		Object i: 1	Object i: 0	
Object j	1	a	b	$a+b$
	0	c	d	$c+d$
		$a+c$	$b+d$	p

- Dice. $S_{dice}(X,Y) = \dfrac{2a}{2a+b+c}$
- Jaccard. $S_{jaccard}(X,Y) = \dfrac{a}{a+b+c}$
- Overlap. $S_{overlap}(X,Y) = \dfrac{a}{min(a+b, a+c)}$
- Cosine. $S_{overlap}(X,Y) = \dfrac{a}{\sqrt{(a+b)(a+c)}}$
- Matching coefficient. $S_{mc}(i,j) = \dfrac{a+d}{p}$

Distributional Similarity

- Particular case of vectorial representation where attributes are probability distributions

$$\vec{x}^T = [x_1 \ldots x_N] \text{ such that } \forall i, 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^{N} x_i = 1$$

- Kullback-Leibler Divergence (Relative Entropy)

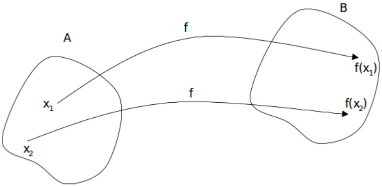
$$D(q||r) = \sum_{y \in Y} q(y) \log \frac{q(y)}{r(y)} \qquad \text{(non symmetrical)}$$

- Mutual Information

$$I(A, B) = D(h||f \cdot g) = \sum_{a \in A} \sum_{b \in B} h(a, b) \log \frac{h(a, b)}{f(a) \cdot g(b)}$$

(KL-divergence between joint and product distribution)

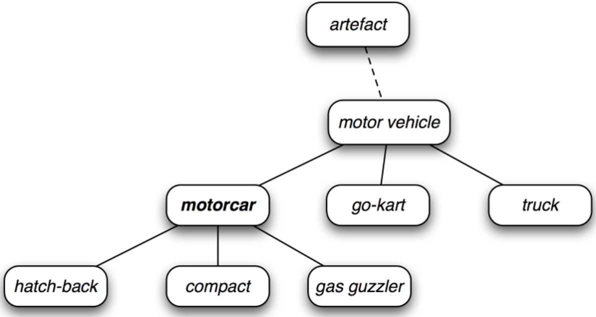
Semantic Similarity



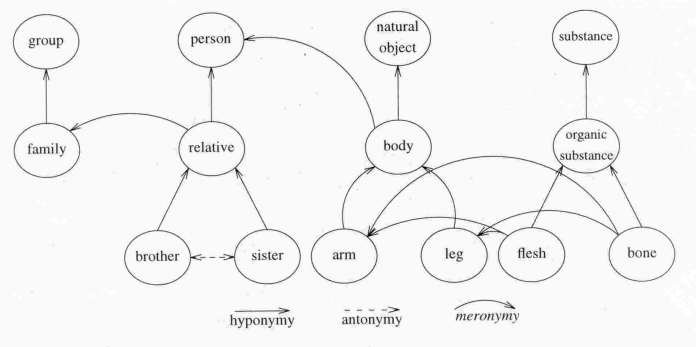
Project objects onto a semantic space:
$D_A(x_1, x_2) = D_B(f(x_1), f(x_2))$

- Semantic spaces: ontology (WordNet, CYC, SUMO, ...) or graph-like knowledge base (e.g. Wikipedia).
- Not easy to project words, since semantic space is composed of concepts, and a word may map to more than one concept.
- Not obvious how to compute distance in the semantic space.

WordNet


artefact
motor vehicle
motorcar
go-kart
truck
hatch-back
compact
gas guzzler

WordNet


group
person
natural object
substance
family
relative
body
organic substance
brother
sister
arm
leg
flesh
bone
hyponymy
antonymy
meronymy

Distances in WordNet

WordNet::Similarity

`http://maraca.d.umn.edu/cgi-bin/similarity/similarity.cgi`

Some definitions:

- $SLP(s_1, s_2)$ = Shortest Path Length from concept s_1 to s_2 (Which subset of arcs are used? antonymy, gloss, ...)
- $depth(s)$ = Depth of concept s in the ontology
- $MaxDepth = \max_{s \in WN} depth(s)$
- $LCS(s_1, s_2)$ = Lowest Common Subsumer of s_1 and s_2
- $IC(s) = -log \frac{1}{P(s)}$ = Information Content of s (given a corpus)

Distances in WordNet

- Shortest Path Length: $D(s_1, s_2) = SLP(s_1, s_2)$
- Leacock & Chodorow: $D(s_1, s_2) = -log \frac{SLP(s_1, s_2)}{2 \cdot MaxDepth}$
- Wu & Palmer: $D(s_1, s_2) = \frac{2 \cdot depth(LCS(s_1, s_2))}{depth(s_1) + depth(s_2)}$
- Resnik: $D(s_1, s_2) = IC(LCS(s_1, s_2))$
- Jiang & Conrath: $D(s_1, s_2) = IC(s_1) + IC(s_2) - 2 \cdot IC(LCS(s_1, s_2))$
- Lin: $D(s_1, s_2) = \frac{2 \cdot IC(LCS(s_1, s_2))}{IC(s_1) + IC(s_2)}$
- Gloss overlap: Sum of squares of lengths of word overlaps between glosses
- Gloss vector: Cosine of second-order co-occurrence vectors of glosses

Distances in Wikipedia

- Measures using links, including measures used on WordNet, but applied to Wikipedia graph

  http://www.h-its.org/english/research/nlp/download/wikipediasimilarity.php
- Measures using content of articles (vector spaces)
- Measures using Wikipedia Categories

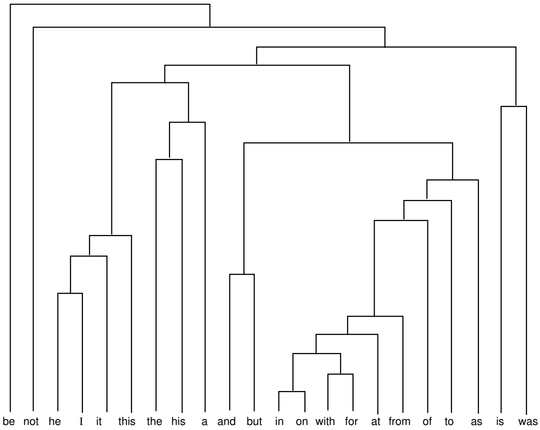
Clustering

- Partition a set of objects into clusters.
- Objects: features and values
- Similarity measure
- Utilities:
  - Exploratory Data Analysis (EDA).
  - Generalization (*learning*). Ex: *on Monday*, *on Sunday*, *? Friday*
- Supervised *vs* unsupervised classification
- Object assignment to clusters
  - Hard. *one cluster per object.*
  - Soft. *distribution* $P(c_i \mid x_j)$. *Degree of membership.*

Clustering

- Produced structures
  - Hierarchical (set of clusters + relationships)
    - Good for detailed data analysis
    - Provides more information
    - Less efficient
    - No single best algorithm
  - Flat / Non-hierarchical (set of clusters)
    - Preferable if efficiency is required or large data sets
    - K-means: Simple method, sufficient starting point.
    - K-means assumes euclidean space, if is not the case, EM may be used.
- Cluster representative
  - Centroid $\vec{\mu} = \frac{1}{|c|} \sum_{\vec{x} \in c} \vec{x}$

Dendogram



Single-link clustering of 22 frequent English words represented as a dendogram.

Hierarchical Clustering

- Bottom-up (Agglomerative Clustering)
  Start with individual objects, iteratively group the most similar.
- Top-down (Divisive Clustering)
  Start with all the objects, iteratively divide them maximizing within-group similarity.

Agglomerative Clustering (Bottom-up)

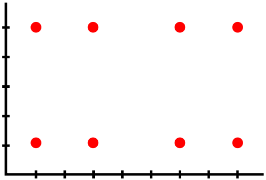
Input: A set $\mathcal{X} = \{x_1, \ldots, x_n\}$ of objects
A function sim: $\mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \longrightarrow \mathcal{R}$
Output: A cluster hierarchy

for $i{:=}1$ **to** n **do** $c_i{:=}\{x_i\}$ **end**
$C{:=}\{c_1, \ldots, c_n\};\ \ j{:=}n+1$
while $C > 1$ **do**
  $(c_{n_1}, c_{n_2}){:=}\arg\max_{(c_u,c_v)\in C\times C} \text{sim}(c_u, c_v)$
  $c_j = c_{n_1} \cup c_{n_2}$
  $C{:=}C \setminus \{c_{n_1}, c_{n_2}\} \cup \{c_j\}$
  $j{:=}j+1$
end–while

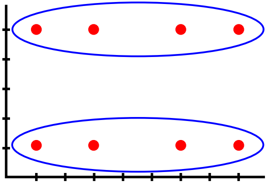
Cluster Similarity

- Single link: Similarity of two most similar members
  - Local coherence (close objects are in the same cluster)
  - Elongated clusters (chaining effect)
- Complete link: Similarity of two least similar members
  - Global coherence, avoids elongated clusters
  - Better (?) clusters
- UPGMA: Unweighted Pair Group Method with Arithmetic Mean
  - $\frac{1}{|X| \cdot |Y|} \sum_{x \in X} \sum_{y \in Y} D(x, y)$
  - Average pairwise similarity between members
  - Trade-off between global coherence and efficiency

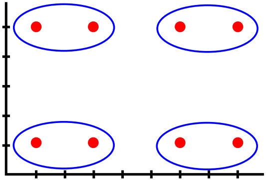
Examples



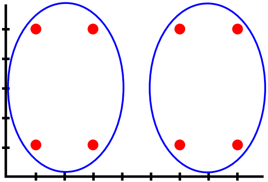
A cloud of points in a plane



Single-link clustering



Intermediate clustering



Complete-link clustering

Divisive Clustering (Top-down)

Input: A set $\mathcal{X} = \{x_1, \ldots, x_n\}$ of objects
A function coh: $\mathcal{P}(\mathcal{X}) \longrightarrow \mathcal{R}$
A function split: $\mathcal{P}(\mathcal{X}) \longrightarrow \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X})$
Output: A cluster hierarchy

$C{:=}\{\mathcal{X}\};\ \ c_1{:=}\mathcal{X};\ \ j{:=}1$
while $\exists c_i \in C$ s.t. $|c_i| > 1$ **do**
$c_u{:=}\arg\min_{c_v \in C}\ \mathrm{coh}(c_v)$
$(c_{j+1}, c_{j+2}) = \mathrm{split}(c_u)$
$C{:=}C \setminus \{c_u\} \cup \{c_{j+1}, c_{j+2}\}$
$j{:=}j+2$
end–while

Top-down clustering

- Cluster splitting: Finding two sub-clusters
- Split clusters with lower *coherence*:
  - Single-link, Complete-link, Group-average
  - Splitting is a sub-clustering task:
    - Non-hierarchical clustering
    - Bottom-up clustering
- Example: Distributional noun clustering (Pereira et al., 93)
  - Clustering nouns with similar verb probability distributions
  - KL divergence as distance between distributions

$$D(p||q) = \sum_{x \in X} p(x) \log \frac{p(x)}{q(x)}$$

  - Bottom-up clustering not applicable due to some $q(x) = 0$

Non-hierarchical clustering

- Start with a partition based on random seeds
- Iteratively refine partition by means of *reallocating* objects
- Stop when cluster quality doesn't improve further
  - group-average similarity
  - mutual information between adjacent clusters
  - likelihood of data given cluster model
- Number of desired clusters ?
  - Testing different values
  - Minimum Description Length: the goodness function includes information about the number of clusters

K-means

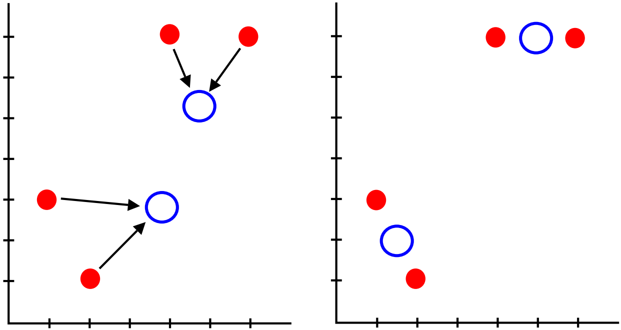
- Clusters are represented by centers of mass (centroids) or a prototypical member (medoid)
- Euclidean distance
- Sensitive to outliers
- Hard clustering
- $\mathcal{O}(n)$

K-means algorithm

Input: A set $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \mathcal{R}^m$
A distance measure $d : \mathcal{R}^m \times \mathcal{R}^m \longrightarrow \mathcal{R}$
A function for computing the mean $\mu : \mathcal{P}(\mathcal{R}) \longrightarrow \mathcal{R}^m$
Output: A partition of $\mathcal{X}$ in clusters

Select k initial centers $\mathbf{f}_1, \ldots, \mathbf{f}_k$
while stopping criterion is not true **do**
    for all clusters c_j **do**
        $c_j := \{\mathbf{x}_i \,|\, \forall\, \mathbf{f}_l \;\; d(\mathbf{x}_i, \mathbf{f}_j) \leq d(\mathbf{x}_i, \mathbf{f}_l)\}$
    for all means $\mathbf{f}_j$ **do**
        $\mathbf{f}_j := \mu(c_j)$
end–while

K-means example



Assignment

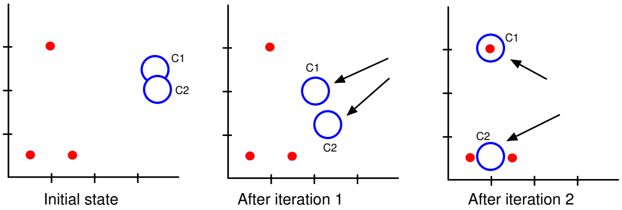
Recomputation of means

EM algorithm

- Estimate the (hidden) parameters of a model given the data
- Estimation–Maximization deadlock
  - Estimation: If we knew the parameters, we could compute the expected values of the hidden structure of the model.
  - Maximization: If we knew the expected values of the hidden structure of the model, we could compute the MLE of the parameters.
- NLP applications
  - Forward-Backward algorithm (Baum-Welch reestimation).
  - Inside-Outside algorithm.
  - Unsupervised WSD

EM example

- Can be seen as a *soft* version of K-means
- Random initial centroids
- Soft assignments
- Recompute (averaged) centroids



An example of using the EM algorithm for soft clustering

Clustering evaluation

- Related to a reference clustering: Purity and Inverse Purity.

$$P = \frac{1}{|D|}\sum_{c} \max_{x} |c \cap x|$$

$$IP = \frac{1}{|D|}\sum_{x} \max_{c} |c \cap x|$$

Where:
c = obtained clusters
x = expected clusters
$|D|$ = number of documents

- Without reference clustering: *Cluster quality* measures: Coherence, average internal distance, average external distance, etc.

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